Semestral Examination
Maximum you can score is 60
Duration: 120 minutes

(1) Let $\mathcal{C}$ be the countable-cocountable σ-field on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that f is $\mathcal{C} - \mathcal{B}_{\mathbb{R}}$-measurable iff it is constant on a cocountable set. **[10]**

(2) Let $(\Omega, \mathfrak{S}, P)$ be a probability space and $\{\mathfrak{S}_n\}_{n\geq 1}$ an increasing sequence of Boolean algebras such that $\mathfrak{S} = \sigma(\mathfrak{S}_n : n \geq 1)$. Show that

$$\forall A \in \mathfrak{S}, \forall \epsilon > 0, \exists B_\epsilon \in \cup_n \mathfrak{S}_n \text{ such that } P(A \triangle B_\epsilon) < \epsilon.$$

[10]

(3) Let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$. Compute $I.I$ in polar coordinates and conclude that $I = \sqrt{2\pi}$. **[10]**

(4) Let X be a random variable having distribution function F. Show that $\mathbb{E}(F(X) \geq 1/2$ with equality iff F is continuous. **[6+2+2]**

(5) Let $\{X_n\}$ be a sequence of random variables with $X_n \sim F_n$, for $n \geq 1$. Let X be a random variable with distribution function F. Let $X_n \xrightarrow{d} X$. Show that $F_n(x_n) \to F(x)$ provided x_n converges to x and x is a continuity point of F.

[10]

(6) Let $(\Omega, \mathfrak{S}, P)$ be a probability space and $\mathfrak{S}' \subseteq \mathfrak{S}$ a sub-sigma-algebra. Let X be a $\mathfrak{S} - \mathcal{B}_{\mathbb{R}}$ measurable integrable map. Define $\mathbb{E}(X|\mathfrak{S}')$. If we have another sigma-algebra $\mathfrak{S}'' \subseteq \mathfrak{S}'$, then show that

$$\mathbb{E}(\mathbb{E}(X|\mathfrak{S}')|\mathfrak{S}'') = \mathbb{E}(X|\mathfrak{S}'').$$

[3+7]

(7) Cleanliness **[2]**